



## H-infinity Optimization of the Recovery Matrix

**Niemann, Hans Henrik; Søgaard-Andersen, P.; Stoustrup, Jakob**

*Published in:*  
Control-Theory And Advanced Technology

*Publication date:*  
1993

*Document Version*  
Publisher's PDF, also known as Version of record

[Link back to DTU Orbit](#)

*Citation (APA):*  
Niemann, H. H., Søgaard-Andersen, P., & Stoustrup, J. (1993). H-infinity Optimization of the Recovery Matrix. *Control-Theory And Advanced Technology*, 9(2), 547-564.

---

### General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Short Paper

## $\mathcal{H}_\infty$ OPTIMIZATION OF THE RECOVERY MATRIX\*

H. H. NIEMANN,<sup>1</sup> P. SØGAARD-ANDERSEN<sup>2</sup> AND J. STOUSTRUP<sup>1</sup>

**Abstract.** The emphasis of this paper is on an alternative approach to the Loop Transfer Recovery (LTR) design problem based on an  $\mathcal{H}_\infty$  optimization of a certain matrix, the Recovery Matrix. The proposed  $\mathcal{H}_\infty$ /LTR method handles both minimum phase systems as well as non-minimum phase systems in a common framework. In both cases, the  $\mathcal{H}_\infty$ /LTR design problem is transformed into an almost disturbance decoupling problem. The resulting controllers are all of the same dynamic order as the plant. As an application, robust control objectives are studied, and sufficient conditions for the robust  $\mathcal{H}_\infty$  problem are given in a form supported by the design method presented in this paper. Furthermore, necessary and sufficient conditions for solvability of the asymptotic recovery problem is given.

**Key Words**—Loop transfer recovery,  $\mathcal{H}_\infty$  theory, robust control, observer-based controllers.

### 1. Introduction

In a series of papers, Doyle and Stein (1979; 1981) introduced the concept of Loop Transfer Recovery (henceforth referred to as LTR) in the control community. In the decade that has elapsed since, the LTR concept has been the focus of numerous studies (Moore and Tay, 1989; Saberi and Sannuti, 1990; Stein and Athans, 1987; Sogaard-Andersen, 1989; Zhang and Freudenberg, 1990).

In the original setting, LTR was intimately related to LQG methods. Later, however, other design methods, such as eigenstructure assignment (Sogaard-Andersen, 1989) and singular perturbation techniques (Saberi and Sannuti, 1990) have also proved to be efficient LTR design methods.

Recently, Moore and Tay (1989) pioneered a new approach to the LTR problem. Their approach is based on an  $\mathcal{H}_\infty$  optimization of a suitably chosen recovery function. This approach is promising in the sense that a more systematic LTR procedure can be devised. In the usual LQG/LTR setting, the LTR design step is highly iterative. A more or less arbitrary design is made, and thereafter tested to see if the specifications have been met. If not, an iterative series of designs is required.  $\mathcal{H}_\infty$  theory, however, offers an appealing alternative. The  $\mathcal{H}_\infty$  design philosophy is to make an advance specification of the  $\mathcal{H}_\infty$  norm, which is subsequently used in the design process. This reduces the need for iterative numerical procedures, since a controller which achieves a specified

\* Received by the editors January 16, 1991 and in finally revised form November 2, 1992.

This work is supported in part by the Danish Technical Research Council, under grant No. 16-4885-2 and grant No. 26-1830-1.

<sup>1</sup> Mathematical Institute, Technical University of Denmark, Building 303, DK-2800 Lyngby, Denmark.

<sup>2</sup> SimCorp A/S, Kompagnistræde 20, DK-1208 København K, Denmark.

$\mathcal{H}_\infty$  norm bound can be designed in a top-down manner. There are still, though, unnecessary iterative steps in the Moore and Tay method. One of the purposes of this paper is to overcome this problem. Another drawback of the Moore and Tay approach is that for systems with right half plane zeros, only the minimum phase part is considered, and, hence, no guaranteed norm bounds for the overall systems can be given. Moreover, the resulting controller order turns out to be at least  $2n$ , which is unnecessarily large, due to the fact that the approach uses frequency domain  $\mathcal{H}_\infty$  techniques, instead of the state space methods which have meanwhile proven more powerful.

In this paper, we propose an alternative approach to the  $\mathcal{H}_\infty$ /LTR problem. The  $\mathcal{H}_\infty$ /LTR design problem is formulated in Sec. 2 as an  $\mathcal{H}_\infty$  optimization of a certain matrix valued function, the Recovery Matrix. We pose the problem in a state space formulation with the Luenberger observer-based approach introduced in Niemann et al. (1991). An  $\mathcal{H}_\infty$  bound for the Recovery Matrix is derived and acts as a sufficient condition for the robust  $\mathcal{H}_\infty$  control problem. Further, we prove constructively that the  $\mathcal{H}_\infty$ /LTR design problem can always be solved by an  $n$ th order controller with a standard full order observer-based controller structure.

A design procedure for the  $n$ th order  $\mathcal{H}_\infty$ /LTR controller is provided in Sec. 3. In this setting, we invoke some recent  $\mathcal{H}_\infty$  results (Stoorvogel, 1992), which directly allow singular systems; i.e., regularity of the direct feedthrough terms is not assumed, as opposed to Moore and Tay (1989), where this restriction is tackled by an approximation technique, which induces an unnecessary iteration parameter.

Necessary and sufficient conditions are given for the solvability of the  $\mathcal{H}_\infty$ /LTR design problem with a specified  $\mathcal{H}_\infty$  norm bound on the recovery matrix.

For minimum phase plants, these results imply simple design algorithms. The designer may specify any level of recovery, and the solution is then a matter of solving linear equations. For systems with RHP zeros, the situation is similar, except that the designer is not allowed to specify an arbitrarily small recovery error. For non-minimum phase systems, the solution involves a state space transformation and a reduced order Riccati equation. By means of these two operations, the problem is thereafter transformed into an  $\mathcal{H}_\infty$ /LTR problem for a minimum phase problem.

Originally, LTR was introduced as an approach to solve the mixed sensitivity problem. However, the methods known for the mixed sensitivity problem at this point are all based on sufficient conditions; i.e., they are inherently conservative. In this paper, we compare various approaches to the mixed sensitivity problem and provide bounds for their conservatism. The method introduced here is conservative as well, since it considers only a common factor of various transfer functions commonly chosen as objects of optimization. However, in Sec. 4, we shall give an example of robust controller design arising from these different  $\mathcal{H}_\infty$ /LTR methods showing that our methods are sometimes less conservative than the known methods, even though the controller order is lower.

## 2. Robust Control and the Recovery Principle

The main issue in the robust control paradigm is to design dynamic feedback compensators which optimally track a reference signal in the face of various kinds of uncertainties, which are usually thought of as e.g., disturbances,

measurement noise, unmodeled dynamics or unknown future reference signals to be applied.

It is well known (Chiang and Safonov, 1988) that all these types of uncertainties relate to two transfer functions, the sensitivity function  $S(s)$  and the complementary sensitivity function  $T(s)$ , given by

$$S(s) = (I - Q(s)G(s))^{-1}, \quad T(s) = (I - (Q(s)G(s))^{-1})^{-1}, \quad (2.1)$$

where  $G(s)$  is the plant transfer function and  $Q(s)$  is the controller to be designed.  $S(s)$  has to be small to suppress disturbances and obtain good tracking. On the other hand, to achieve robust stability subject to unmodeled dynamics and to eliminate the influence of measurement noise,  $T(s)$  has to be small. By introducing weighting functions expressing the *a priori* knowledge of the frequency contents of disturbances and reference signals,  $W_1(\cdot)$ , and the frequency contents of measurement noise and unmodeled dynamics,  $W_2(\cdot)$ , we are led to the following problem formulation.

**Problem 2.1.** Let  $\gamma > 0$  be given, and let  $W_1(\cdot)$  and  $W_2(\cdot)$  be weight functions for the sensitivity function and the complementary sensitivity function, respectively. Find, if possible, a finite dimensional, linear, time invariant, internally stabilizing controller  $Q(s)$ , such that the resulting sensitivity and complementary sensitivity functions satisfy

$$\|W_1(\cdot)S(\cdot)\|_\infty < \gamma \quad \text{and} \quad \|W_2(\cdot)T(\cdot)\|_\infty < \gamma. \quad (2.2)$$

Problem 2.1 is an optimization problem for which there is no direct translation to an  $\mathcal{H}_\infty$  standard problem. Hence, to obtain feasible solutions by means of an  $\mathcal{H}_\infty$  standard model, we have to introduce some amount of conservatism. In this section, we shall discuss several approaches to perform this  $\mathcal{H}_\infty$  standard problem modeling in a Loop Transfer Recovery (LTR) setting (see Stoustrup and Niemann, 1993).

Let us consider a finite dimensional, linear, time invariant plant model, represented by a state space realization  $(A, B, C, 0)$ ,

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu, & x \in \mathbb{R}^n, u \in \mathbb{R}^m, \\ z = Cx, & z \in \mathbb{R}^p. \end{cases} \quad (2.3)$$

To design a controller for a given system by the LTR methodology, we first determine a formal (static) state feedback  $u = Fx$ , the target design, which satisfies our design specifications, in this case

$$\|W_1(\cdot)S_F(\cdot)\|_\infty < \gamma \quad \text{and} \quad \|W_2(\cdot)T_F(\cdot)\|_\infty < \gamma, \quad (2.4)$$

where

$$S_F(s) = (I - F(sI - A)^{-1}B)^{-1} \quad \text{and} \quad T_F(s) = -F(sI - A - BF)^{-1}B \quad (2.5)$$

are the sensitivity function and the complementary sensitivity function, respectively, obtained when applying a state feedback. The target design can be done



in a number of ways, which will not be discussed further in the present paper. Subsequently, the recovery step of the LTR design procedure has to recover the target design sufficiently well by means of a dynamic measurement feedback controller. In the standard LQG/LTR setting (Doyle and Stein, 1979; 1981), only full order observer-based controllers, i.e., a controller  $u = Q(s)y$ , where  $Q$  has a realization of the form,

$$\Sigma_Q: \begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K(C\hat{x} - y) \\ u = F\hat{x} \end{cases} \quad (2.6)$$

are considered. This means that we only have to design an observer gain  $K$ , such that when applying the corresponding full order observer-based controller with feedback gain  $F$ , (2.2) is still satisfied. In this paper, it will be shown that the full order observer-based controller structure can be imposed also in the  $\mathcal{H}_\infty$ /LTR case with good results compared to higher order controllers (see the discussion below).

The first approach to formulate Problem 2.1 as an  $\mathcal{H}_\infty$  standard problem in the LTR setting was taken by Moore and Tay (1989). They applied a frequency domain method to an  $\mathcal{H}_\infty$  optimization problem based on the *recovery error*  $E(s)$ ,

$$E(s) = S(s) - S_F(s) = T_F(s) - T(s) \quad (2.7)$$

(where  $S + T \equiv I$  is exploited).

The motivation for studying an  $\mathcal{H}_\infty$  norm bound on  $E(s)$ , which is not explicitly mentioned in Moore and Tay (1989), is the following. When the actual sensitivity functions  $S(s)$  and  $T(s)$  are sufficiently close to the target sensitivity functions  $S_F(s)$  and  $T_F(s)$ , respectively, so will (2.2) be satisfied when (2.4) is. To be more precise, we require the relations

$$\|W_1(\cdot)S(\cdot)\|_\infty < \gamma, \quad \|W_2(\cdot)T(\cdot)\|_\infty < \gamma, \quad (2.8)$$

which are satisfied, if and only if

$$\|W_1(\cdot)(E(\cdot) + S_F(\cdot))\|_\infty < \gamma, \quad \|W_2(\cdot)(T_F(\cdot) - E(\cdot))\|_\infty < \gamma, \quad (2.9)$$

which in turn is implied by

$$\left. \begin{aligned} \|W_1(\cdot)E(\cdot)\|_\infty + \|W_1(\cdot)S_F(\cdot)\|_\infty &< \gamma \\ \|W_2(\cdot)E(\cdot)\|_\infty + \|W_2(\cdot)T_F(\cdot)\|_\infty &< \gamma \end{aligned} \right\}. \quad (2.10)$$

Hence, (2.2) is guaranteed, if  $E(s)$  satisfies the bound in the following problem.

**Problem 2.2.** Let  $\gamma > 0$  be given. Find, if possible, a finite dimensional, linear, time invariant, internally stabilizing controller  $Q(s)$ , such that the resulting recovery error  $E(s)$  satisfies

$$\|E(\cdot)\|_\infty < \min \left\{ \frac{\gamma - \|W_1(\cdot)S_F(\cdot)\|_\infty}{\|W_1(\cdot)\|_\infty}, \frac{\gamma - \|W_2(\cdot)T_F(\cdot)\|_\infty}{\|W_2(\cdot)\|_\infty} \right\}. \quad (2.11)$$

In Moore and Tay (1989), an  $\mathcal{H}_\infty$  problem with an optimization constraint on  $E(s)$  is studied in frequency domain, giving rise to controllers of order  $3n - 1$  or  $2n$  for square systems. State space formulae for the solution to Problem 2.2 were given in Stoustrup and Niemann (1993) with controller orders of, at most,  $2n$ .

In Sec. 4, we shall study an example, which demonstrates that the bound given by (2.11) is rather conservative. The reason is that we consider an unweighted optimization of  $E(s)$ , which means that we have to accept a low frequency sensitivity error of the same magnitude as the unavoidable error caused by non-minimum phase zeros. This makes the performance specifications very conservative.

To avoid the conservatism of Problem 2.2, we have to introduce the two weights  $W_1(s)$  and  $W_2(s)$  for  $E(s)$ . This gives rise to the following problem.

**Problem 2.3.** Let  $\gamma > 0$  be given. Find, if possible, a finite dimensional, linear, time invariant, internally stabilizing controller  $Q(s)$ , such that the resulting recovery error  $E(\cdot)$  satisfies

$$\left\| \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} E(\cdot) \right\|_\infty < \min \{ \gamma - \| W_1(\cdot) S_F(\cdot) \|_\infty, \gamma - \| W_2(\cdot) T_F(s) \|_\infty \}. \quad (2.12)$$

As in Problem 2.2, the right hand side of (2.12) is an upper bound for the  $\mathcal{H}_\infty$  optimization of the weighted sensitivity recovery error, which guarantees that a controller solving Problem 2.3 also is a solution of Problem 2.1. Problem 2.3 is a sensible approach to a robust  $\mathcal{H}_\infty$  design method in the LTR methodology with regard to conservatism. Provided the target feedback has been carefully selected, the design will be no more conservative than 6 [dB]. On the other hand, a conservatism of 6 [dB] is inevitable, when Problem 2.1 is modeled as an  $\mathcal{H}_\infty$  standard problem. The solution to Problem 2.3 is given in Niemann and Stoustrup (1991).

The drawback of applying an  $\mathcal{H}_\infty$ /LTR method based on Problem 2.3 is associated with controller orders. Since the dynamic order of  $E(s)$  is  $2n$ , a controller for Problem 2.3 will, in the general case, be of order  $2n + n_{w_1} + n_{w_2}$ , where  $n_{w_1}$  and  $n_{w_2}$  are the dynamic orders of  $W_1(s)$  and  $W_2(s)$ , respectively.

The method proposed in this paper will always yield controllers of order at most  $n$ . Before introducing the associated  $\mathcal{H}_\infty$  standard problem, we first need a preliminary observation. The recovery error  $E(s)$  for a feedback system with a full order observer-based controller of the form (2.6) can be written as the following product (Niemann et al., 1991):

$$E(s) = S_F(s)M(s), \quad (2.13)$$

where  $S_F(s)$  is given by (2.5) and the *Recovery Matrix*  $M(s)$  is given by

$$M(s) = F(sI - A - KC)^{-1}B \quad (2.14)$$

and  $K$  is here the observer gain to be designed. Since  $S_F(s)$  is independent of  $K$ , clearly  $\|E(\cdot)\|_\infty \rightarrow 0$ , if and only if  $\|M(\cdot)\|_\infty \rightarrow 0$  for a sequence of controllers. Hence, for the minimum phase case, considering  $\mathcal{H}_\infty$  bounds on  $M(\cdot)$  is equivalent to considering  $\mathcal{H}_\infty$  bounds on  $E(\cdot)$ . In the general non-minimum

phase case, (2.2) leads to the following  $\mathcal{H}_\infty$  constraint on the recovery matrix.

**Problem 2.4.** Let  $\gamma > 0$  be given. Find, if possible, a finite dimensional, linear, time invariant, internally stabilizing controller  $Q(s)$ , such that the resulting recovery matrix  $M(\cdot)$  satisfies

$$\|M(\cdot)\|_\infty < \min \left\{ \frac{\gamma - \|W_1(\cdot)S_F(\cdot)\|_\infty}{\|W_1(\cdot)S_F(\cdot)\|_\infty}, \frac{\gamma - \|W_2(\cdot)T_F(\cdot)\|_\infty}{\|W_2(\cdot)S_F(\cdot)\|_\infty} \right\}. \quad (2.15)$$

This  $\mathcal{H}_\infty$  constraint constitutes a sufficient condition for a solution of Problem 2.4 to be a solution of Problem 2.1, also. In Sec. 4, however, we shall show that sometimes Problem 2.4 has a solution, although Problem 2.2 has not, meaning that Problem 2.2 has a more conservative bound than Problem 2.4 in some cases. The reason is that for typical designs, the “min” of (2.11) and of (2.15) are determined by the first operand in both cases. This means that if  $\Delta = 20 * (\log \|W_1(\cdot)S_F(\cdot)\|_\infty - \log \|W_1(\cdot)\|_\infty)$  is a positive number, which it will always be for reasonable target designs, then an  $\mathcal{H}_\infty$ /LTR method based on Problem 2.4 will be  $\Delta$  [dB] less conservative than a method based on Problem 2.2. Moreover, since the dynamic order of  $M(s)$  is  $n$ , the solution to Problem 2.4 will be a controller of order, at most  $n$ , and hence only half the number of controller states, as in Problem 2.2, is required.

For the purposes in this paper, the recovery matrix function  $M(s)$  is defined in terms of full order observer-based controllers. It can be proven, however, that (2.13) holds for any controller type (see Niemann et al., 1991), and (2.13) can hence be taken as the defining equation.

Let us consider for a moment the more general Luenberger observer-based controller,

$$\Sigma_L: \begin{cases} \dot{\xi} = D\xi + Gu + Ey, \\ u = P\xi + Vy, \end{cases} \quad (2.16)$$

where  $D, G, E, P$  and  $V$  satisfy the Luenberger conditions (Luenberger, 1971),

$$\left. \begin{aligned} \Lambda(D) &\subset \mathbb{C}^- \\ TA - DT &= EC \\ G &= TB \\ F &= PT + VC \end{aligned} \right\}, \quad (2.17)$$

for some matrix  $T$ . The recovery matrix for the Luenberger observer based controller is given by

$$M_L(s) = P(sI - D)^{-1}G. \quad (2.18)$$

It is possible to implement any observer-based controller in the Luenberger architecture (Niemann et al., 1991). Because of this generality, it is relevant to use the Luenberger observer in LTR design, instead of restricting the attention to specific observer types as e.g., full order or minimal order observer-based controllers.

However, for the LTR design problem considered in this paper, i.e., Problem

2.4, it turns out that it contains no loss of generality to consider only full order observer-based controllers as stated in the following result.

**Theorem 2.5.** Let  $\gamma > 0$  be given. The following two statements are equivalent:

- (1) There exists an internally stabilizing Luenberger observer-based controller, such that the resulting recovery matrix  $M_L(\cdot)$  satisfies  $\|M_L(\cdot)\|_\infty < \gamma$ .
- (2) There exists an internally stabilizing full order observer-based controller, such that the resulting recovery matrix  $M(\cdot)$  satisfies  $\|M(\cdot)\|_\infty < \gamma$ .

*Proof.* See Appendix A.

Theorem 2.5 shows that if Problem 2.4 is solvable by any observer-based controller, then it is solvable also by a full order observer-based controller. Using this important observation, we restrict the attention to full order observer-based controllers throughout this paper.

To summarize, we have restated the original  $\mathcal{H}_\infty$  optimization Problem 2.1, which concerns both sensitivity functions, as an  $\mathcal{H}_\infty$  standard problem in three different formulations. Each standard problem formulation is an  $\mathcal{H}_\infty$  optimization of a single term, rather than the two terms of Problem 2.1. During this transformation, we (unavoidably) retain only sufficiency of the solutions relative to Problem 2.1. The sufficiency is stated as the following result, which is verified through trivial calculations.

**Lemma 2.6.** Let  $Q(s)$  be a finite dimensional, linear, time invariant, internally stabilizing controller, satisfying either (2.11), (2.12) or (2.15). Then, (2.2) is also satisfied.

### 3. Solutions to the $\mathcal{H}_\infty$ /LTR Design Problem

Solutions to Problem 2.4 will be derived in this section in terms of design methods for full order observer-based controllers.

The transpose of the recovery matrix (2.14) corresponding to the full order observer has the following standard state space  $\mathcal{H}_\infty$  representation:

$$\Sigma_M^T: \begin{cases} \dot{x} = A^T x + C^T u + F^T w, \\ z = B^T x + 0 \cdot u. \end{cases} \quad (3.1)$$

The  $\mathcal{H}_\infty$ /LTR design problem for a given  $\gamma > 0$  is now, if possible, to design a state feedback control  $u = K^T x$ , which internally stabilizes the plant and makes the  $\mathcal{H}_\infty$  norm of the resulting closed loop transfer function from  $w$  to  $z$ , i.e., the norm of  $M(\cdot)$ , smaller than  $\gamma = \gamma_{\text{LTR}}$ . The matrix  $K$  will then be the observer gain in the resulting full order observer-based controller.

Applying the so-called singular  $\mathcal{H}_\infty$  approach, we shall give a solution to the general non-minimum phase  $\mathcal{H}_\infty$ /LTR problem in Sec. 3.2. As special cases, we shall apply the result to minimum phase systems and pseudo-minimum phase systems, the latter being systems which can be asymptotically recovered for only certain target specifications. In Sec. 3.4, necessary and sufficient conditions for the asymptotic recovery problem will be provided.

**3.1 The singular approach to  $\mathcal{H}_\infty$ /LTR** The  $\mathcal{H}_\infty$  problem with the state

space representation (3.1) is a so-called *singular* problem, because the direct feedthrough term of the  $u \mapsto z$  transfer function does not have full column rank (it is zero) as is required in order to apply the standard regular  $\mathcal{H}_\infty$  theory, as in e.g., Doyle et al. (1989). Instead, the approach of Stoorvogel (1992), which is a generalization of the results from Doyle et al. (1989), will be taken. As a main difference, the singular  $\mathcal{H}_\infty$  approach of Stoorvogel (1992) includes a certain quadratic matrix inequality with some associated rank constraints, rather than the matrix Riccati equation known from Doyle et al. (1989). In our case, the quadratic matrix inequality specializes to a Riccati inequality with three rank constraint, since the system is totally singular; i.e., the direct feedthrough term is zero. Effectively, we have the following result.

**Lemma 3.1.** There exists an internally stabilizing state feedback  $u = K^T x$ , which makes the  $\mathcal{H}_\infty$  norm of  $M(\cdot)$  smaller than  $\gamma > 0$ , if and only if there exists a matrix  $P \geq 0$  satisfying the following four conditions:

$$\left. \begin{array}{ll} \text{(a)} & AP + PA^T + BB^T + \gamma^{-2}PF^T F P \triangleq \tilde{B}\tilde{B}^T \geq 0 \\ \text{(b)} & \text{rank } \tilde{B} = \text{rank } B \\ \text{(c)} & CP = 0 \\ \text{(d)} & (A + \gamma^{-2}PF^T F, \tilde{B}, C) \text{ is a minimum phase system} \end{array} \right\}. \quad (3.2)$$

*Proof.* The four conditions in Lemma 3.1 are obtained from the results in Stoorvogel (1992), when utilizing the fact that the state feedback problem is totally singular.

**3.2 Non-minimum phase systems** When the system  $\Sigma$  considered includes right half plane zeros, the unique matrix  $P$  satisfying Lemma 3.1 (a)–(d) will be nonzero and depend on  $\gamma$ . Moreover, the recovery level  $\gamma$  cannot in general be selected arbitrarily small, thus giving rise to finite recovery errors. For full and minimal order observer architectures, the existence of lower bounds for recovery has been studied in the literature (Niemann and Jannerup, 1989; Stein and Athans, 1987; Zhang and Freudenberg, 1990). However, when  $\gamma$  has been chosen greater than the infimally achievable recovery level, i.e., when Problem 2.3 has any solutions at all, then a full order observer-based controller solving the  $\mathcal{H}_\infty$ /LTR problem can be found by the following procedure. First, a matrix  $P$  is found, satisfying Lemma 3.1 (a)–(d). This step amounts to performing a state space transformation and solving a reduced order matrix Riccati equation (see Stoorvogel, 1992; Stoustrup and Niemann, 1993) for computational details. Secondly,  $K$  is determined as a solution to a dual almost disturbance decoupling problem, which can be solved following the line of Weiland and Willems (1989). When applying a full order observer-based controller with  $K$  as the observer gain to the original plant, we obtain a solution to the  $\mathcal{H}_\infty$ /LTR problem. We have the following result.

**Theorem 3.2.** Assume that  $P \geq 0$  satisfies conditions (a)–(d) of Lemma 3.1. Assume that  $K$  is an output injection which makes  $A + \gamma^{-2}PF^T F + KC$  stable and further satisfies

$$\|F(sI - A - \gamma^{-2}PF^T F - KC)^{-1}\tilde{B}\|_\infty < \gamma. \quad (3.3)$$

Then, a controller  $u = Q(s)y$ , which internally stabilizes the system  $\Sigma_M^T$  and makes the  $\mathcal{H}_\infty$  norm of the closed loop transfer function from  $w$  to  $z$  smaller than  $\gamma$ , is given by

$$\Sigma_Q: \begin{cases} \dot{\xi} = (A + KC)\xi + K \cdot y, \\ u = -F\xi + 0 \cdot y. \end{cases} \quad (3.4)$$

*Proof.* The lemma is obtained from the results in Stoorvogel (1992).

The interpretation of Theorem 3.2 is that the construction of an LTR controller for a non-minimum phase system can be performed as a recovery design problem for a different, transformed plant. The transformed system  $(A + \gamma^{-2}PF^T F, \tilde{B}, C)$  is by construction minimum phase. Hence, the transformed problem is an asymptotic recovery problem, for which explicit solutions are known.

**3.3 Minimum phase systems** It is well known that for minimum phase systems recovery can be achieved arbitrarily well; i.e., asymptotic recovery is always possible. Correspondingly, a sufficient condition for the conditions in Lemma 3.1 to be satisfied for any choice of  $\gamma$ , is that the system  $\Sigma$  considered is minimum phase.

**Lemma 3.3.** Let  $\gamma > 0$  be given. Assume that  $\Sigma$  is a minimum phase system. Then  $P = 0$  is the unique matrix satisfying conditions (a)–(d) of Lemma 3.1.

*Proof.*  $P = 0$  is seen to be a solution for all  $\gamma$ , since the dependency of  $\gamma$  in conditions (a) and (d) vanishes, and  $\tilde{B} = B$ . Hence, everything remaining is just the requirement that  $\Sigma$  is minimum phase, which was assumed. Uniqueness has been proved in Stoorvogel (1992).

Clearly, for a stabilizable and detectable system, the corresponding  $\mathcal{H}_\infty$  problem will be solvable, provided  $\gamma$  has been chosen sufficiently large. Hence, for systems where solvability does not depend on  $\gamma$  at all, the  $\mathcal{H}_\infty$  problem must be solvable for any value of  $\gamma$ . This is the case for minimum phase systems, as seen by the above lemma. For such systems, we are typically faced with the task of designing an infimizing sequence of controllers, which make the  $\mathcal{H}_\infty$  norm tend to zero, rather than just the design of a single controller for a fixed value of  $\gamma$ . To be more specific, the asymptotic recovery problem is the following variant of Problem 2.4.

**Problem 3.4.** Find, if possible, a series of observer gains  $K_\varepsilon$  such that for every  $\varepsilon > 0$  the closed loop system is internally stable, and

$$\|F(sI - A - K_\varepsilon C)^{-1}B\|_\infty < \varepsilon. \quad (3.5)$$

The relationship between  $\varepsilon$  in (3.5) and the original design specifications is again given by the formulas in Sec. 2. To obtain a suitable  $K_\varepsilon$ , it might be convenient to note that (3.5) has the form of a dual almost disturbance decoupling problem; i.e., the problem is to find a stabilizing state feedback  $\tilde{F} = K_\varepsilon^T$  for the auxiliary system,



$$\Sigma^T: \begin{cases} \dot{x} = A^T x + C^T u + F^T w, \\ z = B^T x + 0 \cdot u. \end{cases} \quad (3.6)$$

which makes the  $\mathcal{H}_\infty$  norm of the transfer function from  $w$  to  $z$  smaller than  $\varepsilon$ . This can be done constructively, using e.g., the approach of Weiland and Willems (1989). Hence, one interpretation of Lemma 3.3 is that for a stabilizable and detectable minimum phase system, almost disturbance decoupling is always possible. The design of full order observer-based controllers for minimum phase systems is well studied (Stein and Athans, 1987; Sogaard-Andersen, 1989), so we shall not elaborate any further on this here.

**3.4 Pseudo-minimum phase systems** In Sec. 3.3, it was demonstrated that asymptotic recovery is always possible for minimum phase systems. The minimum phase condition, however, is *not* necessary for asymptotic recovery. To provide necessary and sufficient conditions for solvability of the asymptotic recovery problem, we shall use some notions from geometric control theory. It turns out that the relevant subspace for our problem is the infimal  $\mathcal{L}_1$ -almost detectability subspace, which is normally denoted by  $\mathcal{N}_{b,g}^*$  (Weiland and Willems, 1989). This subspace has the interpretation that it is the smallest subspace with the property that the factor system modulo  $\mathcal{N}_{b,g}^*$  is minimum phase. Algorithms for determining  $\mathcal{N}_{b,g}^*$  can be found in Appendix B.

**Theorem 3.5.** Consider a system  $(A, B, C)$  with a target state feedback  $F$ . Asymptotic recovery is possible, if and only if  $F$  satisfies

$$F \cdot \mathcal{N}_{b,g}^* = \mathcal{C}. \quad (3.7)$$

*Proof.* The theorem is proved in Appendix B.

To check the necessary and sufficient condition of Theorem 3.5, one computes  $\mathcal{N}_{b,g}^*$  using the algorithms (B.1). Note that  $\mathcal{N}_{b,g}^*$  is independent of  $F$ . Hence, it is possible to design a state feedback which is guaranteed to be asymptotic recoverable by allowing only for partial state feedback. To be more specific, assume that in the basis for the state space representation, the first  $\dim(\mathcal{N}_{b,g}^*)$  basis vectors span  $\mathcal{N}_{b,g}^*$ . Now, the target feedback can be recovered, if and only if it has the form  $F = [0 \ F_2]$ ; i.e., the problem of designing a recoverable target state feedback is equivalent to the formal design of a static output feedback  $u = F_2 y$ , with the fictitious measurement  $y = [0 \ I]x$  (when the system is given in the above basis).

The necessary and sufficient condition (3.7) for achieving asymptotic recovery can be interpreted as a requirement that the kernel structure associated with right half plane zeros of the triple  $(A, B, F)$  coincides with that of  $(A, B, C)$ . In the scalar case, this reduces to the condition that the RHP zeros of  $(A, B, F)$  coincide with the RHP zeros of  $(A, B, C)$ , in which case the RHP zeros of  $M(\cdot)$  cancel. Accordingly, we shall refer to (3.7) as the *pseudo-minimum phase condition*. The pseudo-minimum phase condition is only satisfied for non-minimum phase systems, if  $F$  is chosen properly, as it is seen by the following corollary.

**Corollary 3.6.** Let a system  $(A, B, C)$  be given. Then asymptotic recovery is possible for all stabilizing  $F$ , if and only if  $(A, B, C)$  is minimum phase.

*Proof.* By Theorem 3.5, asymptotic recovery is possible for all stabilizing  $F$ , if and only if  $F \cdot \mathcal{N}_{b,g}^* = \epsilon$  for all stabilizing  $F$ . This is equivalent to  $\mathcal{N}_{b,g}^* = \epsilon$ , meaning that  $(A, B, C)$  has no RHP zeros.

#### 4. A Non-minimum Phase Example

In this section, we shall study an example of a non-minimum phase system, for which we shall compare design methods for achieving robust stability and nominal performance based on the solutions to Problem 2.2 and Problem 2.4, respectively. The latter is the subject of this paper, and the first is treated in Stoustrup and Niemann (1993).

The steps in the applied  $\mathcal{H}_\infty$ /LTR algorithms will not be studied in detail. For a more thorough exposition of the methods, please refer to Stoustrup and Niemann (1993). The algorithms for determining  $\mathcal{H}_\infty$ /LTR controller parameters have been implemented as short MATLAB programs, which are available on request to the authors.

Consider the nominal system given by

$$A = \begin{bmatrix} -0.80 & -0.37 & -0.05 \\ 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = [0 \quad 1 \quad -5],$$

which has an RHP zero in  $z = 5$ .

As the target feedback, let us take

$$F = [-5.20 \quad -17.6925 \quad -23.2313].$$

The closed loop system matrix  $A + BF$  has the eigenvalues

$$-2.5, \quad -1.75 \pm 2.5i.$$

The target feedback loop satisfies the performance and robust stability conditions in (2.4) for the following weights:

$$W_1(s) = 0.233 \cdot \frac{(s + 1.5)^3}{(s + 0.17)^3}, \quad W_2(s) = 0.3$$

for  $\gamma = 1.0$ .

Now, let us apply the two  $\mathcal{H}_\infty$ /LTR controller design methods based on the formulations in Problem 2.2 and Problem 2.4, respectively, to this non-minimum phase system. First, we consider Problem 2.4. The largest permissible  $\mathcal{H}_\infty$  norm of the recovery matrix, for which robust stability/nominal performance is guaranteed by the bound (2.15), is given by

$$\|M(\cdot)\|_\infty < \min\{1.9146, 1.9169\}.$$

By iterative methods, we can compute the infimally achievable  $\mathcal{H}_\infty$  norm of the

recovery matrix, which is  $\gamma_M^* = 1.65 \sim 4.35$  [dB]. One choice for the  $\mathcal{H}_\infty$  optimization bound  $\gamma_M$  of  $M(\cdot)$  is  $\gamma_M = 1.85 \sim 5.24$  [dB]. This is feasible, since  $1.65 < \gamma_M < 1.91$ .

An observer gain which achieves  $\|M(\cdot)\|_\infty < 1.85$  is given by

$$K^T = [131.43 \quad 30.20 \quad 6.09] \cdot 10^3.$$

The sensitivity and complementary sensitivity functions for this design and for the target design are shown in Figs. 1 and 2, along with the design specification bounds; i.e.,  $\|W_1(s)\|^{-1}$  and  $\|W_2(s)\|^{-1}$ .

Next, let us for the same example turn to the formulation in Problem 2.2, where we consider the sensitivity recovery error  $E(s)$ , as has been done in Stoustrup and Niemann (1993).

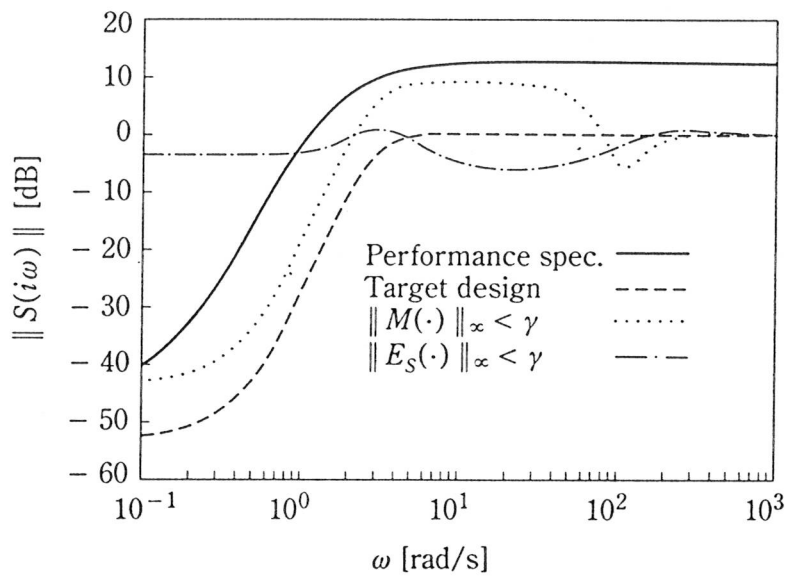


Fig. 1. The sensitivity function.

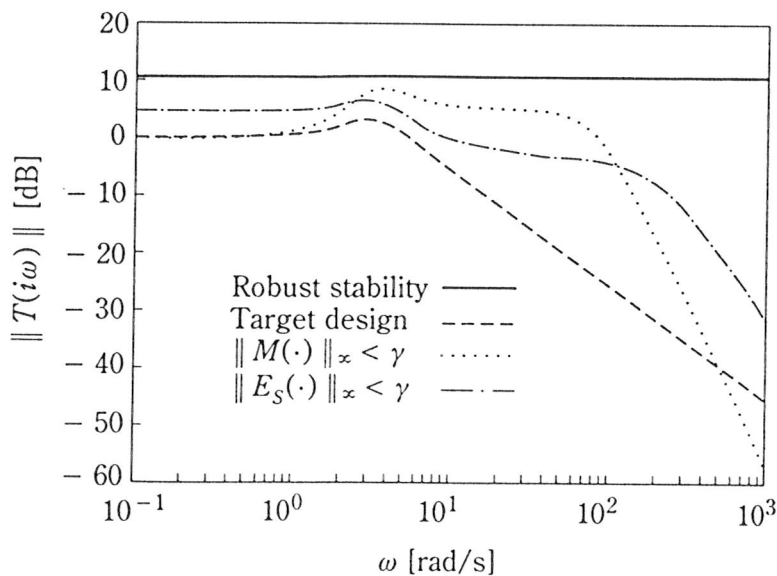


Fig. 2. The complementary sensitivity function.

The upper bound on  $\|E(\cdot)\|_\infty$  is given by (2.11) as

$$\|E(\cdot)\|_\infty < \min\{0.0058, 2.7109\}.$$

However, since the infimally achievable  $\mathcal{H}_\infty$  norm of the sensitivity recovery error  $E(s)$  (determined by an iteration technique) is  $\gamma_E^* = 0.58 \sim -4.73$  [dB], *no feasible choice of an  $\mathcal{H}_\infty$  constraint for  $E(s)$  exists*. Consequently, no method based on the inequality (2.11) can be guaranteed to satisfy the specifications. For comparison, we have computed the sensitivity function and the complementary sensitivity function resulting from an optimization with the optimization constraint  $\|E(\cdot)\|_\infty < 0.65 \sim -3.74$  [dB]. This design will lead to a controller, since  $\gamma = 0.65 > \gamma_E^* = 0.58$ , but it is not guaranteed that we will meet the specifications. In fact, the design violates the nominal performance specifications, as it is seen in Fig. 2, where, as expected, the sensitivity at low frequencies is significantly worse than for the above design (even though the robust stability is slightly better, see Fig. 1). The specification  $\|W_1(s)S(s)\|$  and  $\|W_2(s)T(s)\|$  are shown in Fig. 3. Note, that  $\|W_1(s)S(s)\|$  for the design based on  $E(s)$  is significantly above the  $\gamma = 1.0 \sim 0.0$  [dB] specification.

Hence, we see that for this example, a design based on  $\mathcal{H}_\infty$  optimization of the recovery matrix  $M(s)$  was less conservative (with other choices for  $W_1$  and  $W_2$  this would not necessarily be true) with respect to nominal performance/robust stability specifications than a design based on the sensitivity recovery error  $E(s)$ , even though the recovery matrix design led to a controller with three less controller states. For the present example the classical LQG/LTR method (Doyle and Stein, 1981), which optimizes the  $\mathcal{H}_2$  norm rather than the  $\mathcal{H}_\infty$  norm, did not meet either of the specifications (the design is not shown in the plots). In the scalar case with one RHP zero, the LQG/LTR method will result in a recovery matrix having its  $\mathcal{H}_\infty$  norm exactly twice the infimum (Niemann and Jannerup, 1989) which is not sufficient for the specifications in this example.

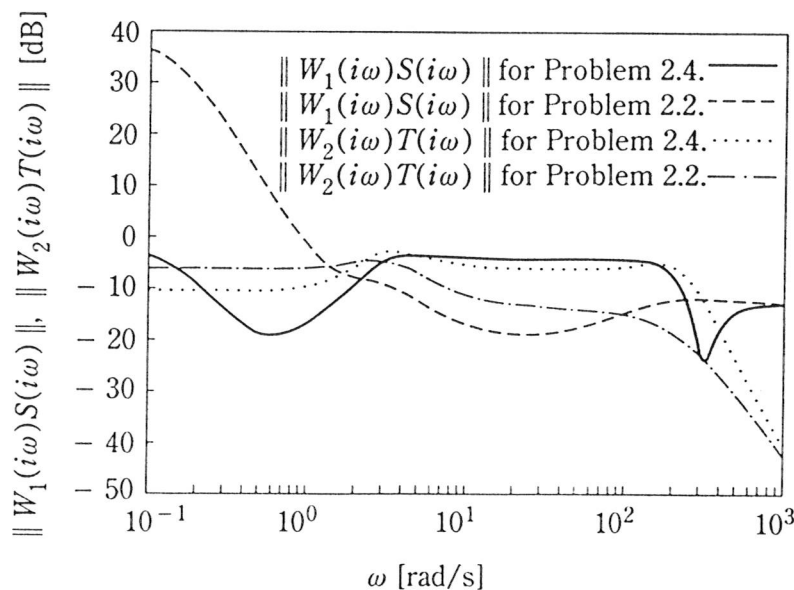


Fig. 3. Weighted sensitivity and complementary sensitivity functions.

## 5. Discussion

Although LTR emerged as an approach to robust control, LQG/LTR does not provide guarantees for robust stability/nominal performance. Hence, the  $\mathcal{H}_\infty$ /LTR design approach proposed in this paper offers a serious alternative to the well known LTR methods for designing robust controllers.

The authors find that this alternative is appealing for a number of reasons. The design method proposed embarks from a prescribed recovery error level. Through a one-shot test procedure, it is determined whether the specifications can be met by a subsequent design procedure or not. This eliminates the need for a full analysis of the closed loop behavior of an iterative sequence of controllers. For (pseudo-) minimum phase systems, the specifications can always be met. For non-minimum phase systems, the synthesis is carried out by means of an auxiliary minimum phase system, which is obtained as a byproduct of the performed test, and the design is eventually carried out exactly as in the minimum phase case. Hence, the method provides a systematic and transparent treatment of non-minimum phase systems, which is only endogenous iterative.

Moreover, in terms of controller order, the method here provides a small controller having only  $n$  controller states, compared to the competing approaches mentioned above which had controller orders of  $2n$ ,  $3n$ . Surprisingly, this does not necessarily mean that more conservatism is introduced, as shown in the design example of Sec. 4, where the present approach based on the recovery matrix was, in fact, less conservative than the sensitivity error based approach, which did not lead to a design satisfying the specifications.

From a conceptual point of view, it is convenient to interpret the  $\mathcal{H}_\infty$ /LTR design problem as disturbance attenuation problems. To be more precise, the exact  $\mathcal{H}_\infty$ /LTR problem (which has not been treated in this paper) is equivalent to an exact disturbance decoupling problem (see Niemann et al., 1991), the asymptotic  $\mathcal{H}_\infty$ /LTR problem is equivalent to an almost disturbance decoupling problem (see Sec. 3.2), and the general non-minimum phase  $\mathcal{H}_\infty$ /LTR problem is equivalent to an  $\mathcal{H}_\infty$  disturbance attenuation problem (i.e., the problem of making the disturbance transfer smaller than a prescribed  $\gamma$ —see Secs. 3.1 and 3.2). The latter is by means of an auxiliary minimum phase system, again interpreted as an almost disturbance decoupling problem. Making stable and efficient algorithms for the almost disturbance decoupling problem therefore becomes the main effort in developing software for the  $\mathcal{H}_\infty$ /LTR problem.

## References

- Chiang, R.Y. and M.G. Safonov (1988). *Robust Control Toolbox—User's Guide*. The Mathworks, Inc., MA.
- Doyle, J. and G. Stein (1979). Robustness with observers. *IEEE Trans. Automatic Control*, **AC-24**, 4, 607–611.
- Doyle, J. and G. Stein (1981). Multivariable feedback design: Concepts for a classical/modern synthesis. *IEEE Trans. Automatic Control*, **AC-26**, 1, 4–16.
- Doyle, J., K. Glover, P. Khargonekar and B.A. Francis (1989). State space solutions to standard  $\mathcal{H}_2$  and  $\mathcal{H}_\infty$  control problems. *IEEE Trans. Automatic Control*, **AC-34**, 8, 831–847.
- Luenberger, D.G. (1971). An introduction to observers. *IEEE Trans. Automatic Control*, **AC-16**, 6, 596–602.

- Moore, J.B. and T.T. Tay (1989). Loop recovery via  $\mathcal{H}_\infty/\mathcal{H}_2$  sensitivity recovery. *Int. J. Control*, **49**, 4, 1249–1271.
- Niemann, H.H. and O. Jannerup (1989). An analysis of pole/zero cancellation in LTR based feedback design. *Proc. American Control Conf. 1990*, San Diego, 848–853.
- Niemann, H.H. and J. Stoustrup (1991). An  $\mathcal{H}_\infty$ /LTR design problem. *Proceedings of the 30th IEEE Conf. on Decision and Control*, Brighton, England, 1920–1926.
- Niemann, H.H., P. Sogaard-Andersen and J. Stoustrup (1991). Loop transfer recovery for general observer architectures. *Int. J. Control*, **53**, 5, 1177–1203.
- Saberi, A. and P. Sannuti (1990). Observer design for loop transfer recovery and for uncertain dynamical systems. *IEEE Trans. Automatic Control*, **AC-35**, 8, 878–897.
- Stein, G. and M. Athans (1987). The LQG/LTR procedure for multivariable feedback control design. *IEEE Trans. Automatic Control*, **AC-32**, 2, 105–114.
- Stoorvogel, A.A. (1992). *The  $\mathcal{H}_\infty$  Control Problem: A State Space Approach*. Prentice-Hall, Englewood Cliffs, N.J.
- Stoustrup, J. (1990). On the solution to a certain Riccati equation appearing in robust control. MAT-Report No. 1990-22, Math. Inst., Tech. Univ. of Denmark.
- Stoustrup, J. and H.H. Niemann (1993). State space solutions to the  $\mathcal{H}_\infty$ /LTR design problem. *Int. J. Robust and Nonlinear Control*, **3**, 1, 1–45.
- Sogaard-Andersen, P. (1989). Loop transfer recovery—An eigenstructure interpretation. *Control—Theory and Advanced Technology (C-TAT)*, **5**, 3, 351–365.
- Weiland, S. and J.C. Willems (1989). Almost disturbance decoupling with internal stability. *IEEE Trans. Automatic Control*, **AC-34**, 3, 277–286.
- Wonham, W.M. (1985). *Linear Multivariable Control—A Geometric Approach*. 3rd ed., Springer-Verlag, N.Y.
- Zhang, Z. and J.S. Freudenberg (1990). Loop transfer recovery for non-minimum phase plants. *IEEE Trans. Automatic Control*, **AC-35**, 5, 547–553.

### Appendix A: Proof of Theorem 2.5

The proof proceeds as follows. In this appendix, we shall prove the equivalence of the existence of a Luenberger observer-based controller to four certain conditions. The same four conditions are shown in Sec. 3 to be equivalent to the existence of a full order observer-based controller. This establishes the theorem.

A complete characterization of all possible recovery matrices obtainable by applying a Luenberger observer-based controller is given by (Niemann et al., 1991) the expression

$$\begin{aligned} M_L(s) &= P(sI - D)^{-1}G \\ &= F(sI - A - KC)^{-1}B + Q(s)C(sI - A - KC)^{-1}B, \end{aligned} \quad (\text{A.1})$$

where  $K$  is an arbitrary but fixed gain such that  $A + KC$  is stable, and  $Q \in \mathcal{RH}_\infty$  is a free parameter. The result is based on the Youla parameterization of all stabilizing controllers. The real rational subclass of these has, in fact, a simple parameterization in the form of Luenberger observer-based controllers (Niemann et al., 1991).

The  $\mathcal{H}_\infty$  optimization  $\|M_L(\cdot)\|_\infty < \gamma$  or equivalently

$$\|F(sI - A - KC)^{-1}B + Q(s)C(sI - A - KC)^{-1}B\|_\infty < \gamma \quad (\text{A.2})$$

becomes a singular model matching problem in the unknown parameter



$Q \in \mathcal{RH}_\infty$ . By applying the singular  $\mathcal{H}_\infty$  theory introduced in Stoorvogel (1992) we get the following conditions for the existence of a  $Q \in \mathcal{RH}_\infty$ , such that  $\|M_L(\cdot)\|_\infty < \gamma$ .

**Lemma A.1.** There exists an internally stabilizing (in terms of the full closed loop system) transfer function  $Q \in \mathcal{RH}_\infty$ , which makes the  $\mathcal{H}_\infty$  norm of  $M_L(\cdot)$  smaller than  $\gamma > 0$ , if and only if there exist matrices  $P_1 \geq 0$  and  $P_2 \geq 0$  satisfying the following seven conditions:

- (1)  $(A + KC)P_2 + P_2(A + KC)^T + BB^T + \gamma^{-2}P_2F^TFP_2 \triangleq \tilde{B}\tilde{B}^T \geq 0$ .
- (2)  $\text{rank } \tilde{B} = \text{rank } B$ .
- (3)  $CP_2 = 0$ .
- (4)  $(A + KC + \gamma^{-2}P_2F^TF, \tilde{B}, C)$  is a minimum phase system.
- (5)  $(A + KC)^TP_1 + P_1(A + KC) + \gamma^{-2}P_1BB^TP_1 = 0$ .
- (6)  $A + KC + \gamma^{-2}BB^TP_1$  is a stability matrix.
- (7)  $\varrho(P_1P_2) < \gamma^2$ .

*Proof.* The 7 conditions in Lemma A.1 follow by applying the main result of Stoorvogel (1992) to the model matching problem (A.2).

By the stability of  $A + KC$ , it follows (Stoustrup, 1990) that for any  $\gamma$ , the unique matrix satisfying (5), (6) and (7) is  $P_1 = 0$ . Hence, an admissible Luenberger observer, i.e., a stabilizing controller, for which  $\|M_L(\cdot)\|_\infty < \gamma$ , exists, if and only if conditions (1)–(4) above are satisfied. Now, in Sec. 3, these four conditions will be shown to be equivalent to the existence of an admissible full order observer-based controller. This establishes Theorem 2.5.

## Appendix B

In Sec. 3.4, necessary and sufficient conditions for asymptotic recovery were given in terms of a certain subspace  $\mathcal{J}_{b,g}^*$ . We shall here provide algorithms for determining  $\mathcal{J}_{b,g}^*$  in the “geometric” style, since matrix notation tends to get unhandy, although the matrix algorithms are relatively simple. Define the subspaces  $\mathcal{B} = \text{Im } B$  and  $\mathcal{K} = \ker C$ . Consider the following recursions:

$$\text{ISA: } \left\{ \begin{array}{l} \mathcal{J}_0 = \mathcal{K} \\ \mathcal{J}_{k+1} = \mathcal{K} \cap A^{-1}(\mathcal{J}_k + \mathcal{B}) \end{array} \right\}, \quad (\text{B.1})$$

and

$$\text{ACSA}_b: \left\{ \begin{array}{l} \mathcal{J}_0 = \mathcal{C} \\ \mathcal{J}_{k+1} = \mathcal{B} + A(\mathcal{K} \cap \mathcal{J}_k) \end{array} \right\}$$

and denote  $\mathcal{A}^* = \mathcal{J}_n$ ,  $\mathcal{J}^* = \mathcal{J}_n$ . (ISA stands for Invariant Subspace Algorithm, ACSA stands for Almost Controllability Subspace Algorithm). Let  $L$  be an arbitrary output injection satisfying  $(A + LC)\mathcal{J}^* \subset \mathcal{J}^*$  (such that  $L$  can be found using the DDP algorithm in Wonham (1985)). Let  $\alpha(\lambda) = \alpha_-(\lambda)\alpha_+(\lambda)$  be the minimal polynomial for  $A + LC$ , where the roots of  $\alpha_-(\lambda)$  lie in  $\mathbb{C}^-$  and those of  $\alpha_+(\lambda)$  lie in  $\bar{\mathbb{C}}^+$ . Define

$\mathcal{X}_-(A + LC) = \text{Im } \alpha_-(A + LC)$ ; i.e.,  $\mathcal{X}_-(A + LC)$  is the orthogonal complement to the generalized stable subspace of  $(A + LC)^T$  (which can be found by calculating the ordered Schur form of  $(A + LC)^T$ ). Then,

$$\mathcal{J}_{b,g}^* = \mathcal{V}^* \cap (\mathcal{J}^* + \mathcal{X}_-(A + LC)). \quad (\text{B.2})$$

Hence, to check the pseudo-minimum phase condition (3.7), a finite recursion of solving linear equations and a Schur decomposition is required.

We now proceed to the proof of Theorem 3.5, which is done by dualization. First, we claim that the dual subspace of  $\mathcal{J}_{b,g}^*$  is  $\gamma_{b,g}^*$ , the supremal  $\mathcal{L}_1$ -almost stabilizability subspace. This is verified by observing that the algorithm given above is the dual of the algorithm given in Weiland and Willems (1989) for determining  $\gamma_{b,g}^*$ . Consider the auxiliary system,

$$\Sigma^T: \begin{cases} \dot{x} = A^T x + C^T u + F^T w, \\ z = B^T x + 0 \cdot u. \end{cases} \quad (\text{B.3})$$

Now, the  $\mathcal{H}_\infty$  almost disturbance decoupling problem is solvable for  $\Sigma^T$ , if and only if the following inclusion holds (see Weiland and Willems, 1989),

$$\text{Im } F^T \subset \gamma_{b,g}^*(\Sigma^T), \quad (\text{B.4})$$

which is equivalent to

$$\ker F \supset \gamma_{b,g}^*(\Sigma^T)^\perp = \mathcal{J}_{b,g}^*(\Sigma), \quad (\text{B.5})$$

$$F \mathcal{J}_{b,g}^*(\Sigma) = \emptyset. \quad (\text{B.6})$$

On the other hand, by definition the  $\mathcal{H}_\infty$ -almost disturbance decoupling problem for  $\Sigma^T$  is solvable, if and only if for all  $\gamma > 0$  there exists  $F_\gamma$ , such that

$$\|B^T(sI - A^T - C^T F_\gamma)^{-1} F^T\|_\infty < \gamma, \quad (\text{B.7})$$

or equivalently

$$\|F(sI - A - F_\gamma^T C)^{-1} B\|_\infty < \gamma, \quad (\text{B.8})$$

which settles the proof by choosing  $K^* = F_\gamma^T$ .





**Hans Henrik Niemann** was born in Slagelse, Denmark, January 4, 1961. He received the M.Sc. degree in mechanical engineering in 1986 and the Ph.D. degree in 1988 from the Technical University of Denmark.

Since 1988 he has had a research position in control engineering at the Technical University of Denmark.

His research interests are: Robust stability and robust performance, Uncertain systems, Robust adaptive control, Robust and reliable control, Computer Aided Control System Design (CACSD), Numerical methods.



**Per Sogaard-Andersen** was born in Holbæk, Denmark, September 23, 1957. He received the M.Sc. in electrical engineering in 1983 and the Ph.D. in 1986 from the Technical University in Denmark. In 1986-87, he held a research position in control engineering at the Technical University. Since 1987 he has been with SimCorp A/S.

His research interests are the application of operations research and stochastic calculus in financial analysis.



**Jakob Stoustrup** was born in Aalborg, Denmark, January 16, 1963. He received the M.Sc. in EE in 1987 and the Ph.D. degree in applied mathematics in 1991, both from the Technical University of Denmark.

Since 1991 he has been Assistant Professor at the Mathematical Institute, Technical University of Denmark.

Currently he is participating in research programs on: *Robust Control for Finite and Infinite Dimensional Systems*, *Partial and Ordinary Differential Equations* and *Robust and Reliable Control*.